Termination of the phase of quintessence by gravitational back reaction

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We study the effects of gravitational back reaction in models of quintessence. The effective energy-momentum tensor with which cosmological fluctuations back react on the background metric will in some cases lead to a termination of the phase of acceleration. The fluctuations we make use of are the perturbations in our present Universe. Their amplitude is normalized by recent measurements of anisotropies in the cosmic microwave background; their slope is taken to be either scale invariant or characterized by a slightly blue tilt. In the latter case, we find that the back-reaction effect of fluctuations whose present wavelength is smaller than the Hubble radius but which are stretched beyond the Hubble radius by the accelerated expansion during the era of quintessence domination can become large. Since the back-reaction effects of these modes oppose acceleration, back reaction will lead to a truncation of the period of quintessence domination. This result impacts on the recent discussions of the potential incompatibility between string theory and quintessence.

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I. INTRODUCTION

Evidence is increasing that the Universe is spatially flat and, at the present time, accelerating. The best evidence for a spatially flat Universe comes from the location of the acoustic peaks of the spectrum of cosmic microwave anisotropies (see e.g. [1] for a recent discussion). Since dynamical mass determinations from observations of large-scale structure are converging [2] to a mass density far short of the critical density, the density required for a spatially flat Universe, the difference must be due to either a remnant small cosmological constant, or a new form of matter which is not clustered gravitationally on the scale of galaxy clusters, and has been given various names, including *dark energy* and *quintessence*, the name we will adopt (see, e.g., [3,4] for original papers and [5] for a recent review and detailed references). This conclusion is supported by the recent supernova observations which yield Hubble diagrams which directly support the evidence that the Universe is, at the present time, accelerating [6,7]. Hence, quintessence must have an equation of state $p<-(1/3)\rho$, p and ρ denoting the pressure and the energy density, respectively.

Most models of quintessence proposed involve a new scalar field Q which is taken to be homogeneous in space, and via its kinetic and potential energy contributions to the energy-momentum tensor $T_{\mu\nu}$ tuned to provide an equation of state leading to accelerated expansion, beginning to dominate the matter content of the Universe today. In many such models, acceleration will continue forever.

It can easily be seen that in such models there is an event horizon for every observer. As has recently been pointed out [8,9], this leads to a potential incompatibility between models of quintessence and string theory. It has been argued (see e.g. [10] and references therein) that in the presence of de Sitter horizons, string theory cannot be defined since physical observables cannot be expressed in terms of the S matrix of string theory. As is obvious [8,9], the problem immediately carries over to models of quintessence postulated which today and, in many models, at all future times, dominates the matter content of the Universe (this conclusion can be avoided in certain models with an exponential potential along "tracker" solutions [11,12]).

By adding more new fields, it is possible to solve (see e.g. [13–15]) this problem by constructing models in which the period of quintessence is terminated in a way which is analogous to the termination of the period of inflation in hybrid inflation [16] models. In this paper, we point out that the gravitational back reaction of cosmological fluctuations provides a mechanism which, in some models, will lead to a termination of the phase of acceleration, without the need to add new physics.

II. METHOD AND QUALITATIVE CONSIDERATIONS

The idea of gravitational back reaction of cosmological fluctuations is simple. In the presence of the fluctuations of the space-time metric and of matter, the cosmological background evolves differently compared to its evolution in the absence of perturbations [17,18]. This effect is due to the nonlinearities of the Einstein field equations. The effect can be characterized in terms of an effective energy-momentum tensor $\tau_{\mu\nu}$ with which the fluctuations back react on the background metric.

In this paper we will focus on the back-reaction effect of infrared modes [modes with wavelength larger than the Hubble radius $H^{-1}(t)$, where $H(t)$ is the Hubble expansion rate] on the evolution of a Universe dominated by a quintessence field. In a separate paper, we will analyze the back-

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reaction effect by ultraviolet modes [19].

In models with accelerated expansion, the phase space of infrared modes grows since the Hubble radius is decreasing in comoving coordinates, as is the case in inflationary cosmology. Hence, we expect the back-reaction effect of infrared modes to grow. In the following we will show that in models with a sufficiently blue spectrum of primordial fluctuations, the back-reaction terms will eventually dominate the field equations for the effective background. Since the equation of state corresponding to τ_{ij} opposes acceleration, back-reaction effects cannot be neglected and are expected to terminate the accelerated expansion, or, in other words, to screen the effects of the background quintessence field.

We first give a brief review of gravitational back reaction of a general scalar matter field φ and then apply it to the period of quintessence domination.

The basic idea of gravitational back reaction (see e.g. [20] for a review) is to expand the Einstein equations to second order in the perturbations, to assume that the first order terms satisfy the equations of motion for linearized cosmological perturbations (see e.g. [21] for a comprehensive review), to take the spatial average of the remaining terms, and to regard the resulting equations as equations for a new homogeneous metric $g_{\mu\nu}^{(0,br)}$ which includes the effect of the perturbations to quadratic order:

$$G_{\mu\nu}(g_{\alpha\beta}^{(0,br)})=8\pi G[T_{\mu\nu}^{(0)}+\tau_{\mu\nu}], \tag{1}$$

where the effective energy-momentum tensor $\tau_{\mu\nu}$ of gravitational back reaction contains the terms resulting from spatial averaging of the second order metric and matter perturbations:

$$\tau_{\mu\nu}=\left\langle T_{\mu\nu}^{(2)}-\frac{1}{8\pi G}G_{\mu\nu}^{(2)}\right\rangle, \tag{2}$$

where the angular brackets stand for spatial averaging, and the superscripts indicate the order in perturbations.

In longitudinal gauge the perturbed metric can be written in the form

$$ds^2=(1+2\Phi)dt^2-a(t)^2(1-2\Phi)\delta_{ij}dx^i dx^j, \tag{3}$$

where $a(t)$ is the cosmological scale factor. Provided there are no linear contributions to the spatial off-diagonal terms in the matter energy-momentum tensor, the above metric contains the full information about scalar metric fluctuations. The energy-momentum tensor for a scalar field is

$$T_{\mu\nu}=\varphi_{,\mu}\varphi_{,\nu}-g_{\mu\nu}\left[\frac{1}{2}\varphi^{,\alpha}\varphi_{,\alpha}-V(\varphi)\right]. \tag{4}$$

In this case, the spatial off-diagonal terms in $T_{\mu\nu}$ vanish.

By expanding the Einstein tensor and the above energy-momentum tensor to second order in the metric and matter fluctuations Φ and $\delta\varphi$, respectively, it can be shown that the non-vanishing components of the effective back-reaction energy-momentum tensor $\tau_{\mu\nu}$ become

$$\tau_{00}=\frac{m_{pl}^2}{8\pi}[+12H\langle\Phi\dot{\Phi}\rangle-3\langle(\dot{\Phi})^2\rangle+9a^{-2}\langle(\nabla\Phi)^2\rangle] + \frac{1}{2}\langle(\delta\dot{\varphi})^2\rangle+\frac{1}{2}a^{-2}\langle(\nabla\delta\varphi)^2\rangle+\frac{1}{2}V''(\varphi_0)\langle\delta\varphi^2\rangle + 2V'(\varphi_0)\langle\Phi\delta\varphi\rangle, \tag{5}$$

and

$$\tau_{ij}=a^2\delta_{ij}\left\{\frac{m_{pl}^2}{8\pi}\left[(24H^2+16\dot{H})\langle\Phi^2\rangle+24H\langle\dot{\Phi}\Phi\rangle+\langle(\dot{\Phi})^2\rangle + 4\langle\Phi\ddot{\Phi}\rangle-\frac{4}{3}a^{-2}\langle(\nabla\Phi)^2\rangle\right]+4\dot{\varphi}_0{}^2\langle\Phi^2\rangle+\frac{1}{2}\langle(\delta\dot{\varphi})^2\rangle -\frac{1}{6}a^{-2}\langle(\nabla\delta\varphi)^2\rangle-4\dot{\varphi}_0\langle\delta\dot{\varphi}\Phi\rangle-\frac{1}{2}V''(\varphi_0)\langle\delta\varphi^2\rangle + 2V'(\varphi_0)\langle\Phi\delta\varphi\rangle\right\}. \tag{6}$$

All terms are quadratic in the fluctuations. The two-point functions in the angular brackets can be viewed classically as spatial averages or quantum mechanically as equal-time two-point functions.

We will now apply these equations to quintessence models and thus replace φ by Q. Since we are interested in the back-reaction effect of infrared modes, we can drop all terms involving spatial gradients. We will focus attention on quintessence models for which the slow-rolling approximation is valid. In this case, the background satisfies

$$\dot{Q}\simeq-\frac{V'}{3H}, \quad H^2\simeq\frac{8\pi}{3m_{pl}^2}V, \quad \frac{\dot{H}}{H^2}\simeq-\frac{m_{pl}^2}{16\pi}\left(\frac{V'}{V}\right)^2, \tag{7}$$

where $V(Q)$ is the potential for the quintessence field, and a prime denotes the derivative with respect to Q. In this case, the expressions for $\tau_{\mu\nu}$ become

$$\rho_{br}\equiv\tau_0^0=\frac{m_{pl}^2}{8\pi}(12H\langle\Phi\dot{\Phi}\rangle-3\langle\dot{\Phi}^2\rangle)+\frac{1}{2}\langle\delta\dot{Q}^2\rangle+\frac{1}{2}V''\langle\delta Q^2\rangle + 2V'\langle\Phi\delta Q\rangle, \tag{8}$$

$$p_{br}\equiv-\frac{1}{3}\tau_i^i=\frac{m_{pl}^2}{8\pi}\bigg(16\dot{H}_Q\langle\Phi^2\rangle+24H\langle\Phi\dot{\Phi}\rangle+\langle\dot{\Phi}^2\rangle+4\langle\Phi\ddot{\Phi}\rangle+\frac{4V'^2}{3V}\langle\Phi^2\rangle\bigg)+8V\langle\Phi^2\rangle+\frac{1}{2}\langle\dot{\delta Q}^2\rangle-\frac{1}{2}V''\langle\delta Q^2\rangle+2V'\langle\Phi\,\delta Q\rangle+\frac{4V'}{3H}\langle\dot{\delta Q}\Phi\rangle. \tag{9}$$

Each two-point function can be evaluated as an integral over the Fourier modes (we assume a flat universe for simplicty) of the linear fluctuation variables. We use the following convention for the Fourier expansion of a function U(**x**,t) (e.g., $\Phi,\delta Q$):

$$U(\mathbf{x},t)=\frac{1}{(2\pi)^{3/2}}\int U_k(t)e^{-i\mathbf{k}\cdot\mathbf{x}}d^3k. \tag{10}$$

Making use of the slow-roll approximation for the quintessence field Q, the linear perturbation equations have the following solution for Φ_k on scales larger than the Hubble radius [21]:

$$\Phi_k\simeq A_k\frac{m_{pl}^2}{16\pi}\left(\frac{V'}{V}\right)^2, \tag{11}$$

where A_k is an integration constant. By taking derivatives of Eq. (11) and using the background equation (7) obtained in the slow-rolling approximation, we can easily find

$$\dot{\Phi}_k\simeq-\frac{2V}{3H_Q}\left[\frac{V''}{V}-\left(\frac{V'}{V}\right)^2\right]\Phi_k,$$
$$\ddot{\Phi}_k\simeq\frac{m_{pl}^2}{24\pi}\left(7\frac{V'^4}{V^3}+2\frac{V'''V'}{V}-13\frac{V'^2V''}{V^2}+4\frac{V''^2}{V}\right)\Phi_k. \tag{12}$$

The linearized Einstein constraint equations yield

$$\delta Q_k=\frac{m_{pl}^2}{4\pi\dot{Q}}(\dot{\Phi}_k+H_Q\Phi_k)=\left[\frac{m_{pl}^2}{2\pi}\left(\frac{V''}{V'}-\frac{V'}{V}\right)-2\frac{V}{V'}\right]\Phi_k, \tag{13}$$

from which the time derivative $\dot{\delta Q}_k$ can easily be expressed in terms of Φ_k, making use once again of the background equations in the slow-rolling approximation.

Inserting Eqs. (10)–(13) into Eqs. (8) and (9), we obtain

$$\rho_{br}=\Bigg\{2V\left(\frac{VV''}{V'^2}-2\right)+\frac{m_{pl}^2}{12\pi}\left(\frac{V'^2}{V}+10V''-11\frac{VV''^2}{V'^2}\right)+\frac{m_{pl}^4}{48\pi^2}\left(2\frac{V'V'''}{V}-3\frac{V''^2}{V}-4\frac{V'^2V''}{V^2}+3\frac{V'^4}{V^3}-2\frac{V''V'''}{V'}+4\frac{V''^3}{V'^2}\right)+\frac{m_{pl}^6}{192\pi^3V}\left(V'''+\frac{V''^2}{V'}-5\frac{V''V'}{V}+3\frac{V'^3}{V^2}\right)^2\Bigg\}\times\langle\Phi^2\rangle, \tag{14}$$

$$p_{br}=\Bigg\{-2V\left(\frac{VV''}{V'^2}-2\right)+\frac{m_{pl}^2}{12\pi}\left(7\frac{V'^2}{V}-22V''+13\frac{VV''^2}{V'^2}\right)+\frac{m_{pl}^4}{48\pi^2}\left(25\frac{V''^2}{V}-17\frac{V'^2V''}{V^2}+2\frac{V'^4}{V^3}-2\frac{V''V'''}{V'}-8\frac{V''^3}{V'^2}\right)+\frac{m_{pl}^6}{192\pi^3V}\left(V'''+\frac{V''^2}{V'}-5\frac{V''V'}{V}+3\frac{V'^3}{V^2}\right)^2\Bigg\}\times\langle\Phi^2\rangle. \tag{15}$$

The energy density and pressure of the background for quintessence during the slow-roll period is

$$\rho_{bg}\simeq-p_{bg}\simeq V, \tag{16}$$

so we have

$$\frac{\rho_{br}+3p_{br}}{\rho_{bg}+3p_{bg}}\simeq\langle\Phi^2\rangle\Bigg\{2\left(\frac{VV''}{V'^2}-2\right)-\frac{m_{pl}^2}{12\pi}\times\left(11\frac{V'^2}{V^2}-28\frac{V''}{V}+14\frac{V''^2}{V'^2}\right)-\frac{m_{pl}^4}{96\pi^2}\times\left(2\frac{V'V'''}{V^2}+72\frac{V''^2}{V^2}-55\frac{V'^2V''}{V^3}+9\frac{V'^4}{V^4}-8\frac{V''V'''}{V'V}-20\frac{V''^3}{V'^2V}\right)-\frac{m_{pl}^6}{96\pi^3V^2}\times\left(V'''+\frac{V''^2}{V'}-5\frac{V''V'}{V}+3\frac{V'^3}{V^2}\right)^2\Bigg\}. \tag{17}$$

When this ratio becomes of order unity, the back-reaction effects begin to dominate. Hence, in the following we will focus on calculating this ratio.

The crucial quantity to evaluate is the expectation value $\langle\Phi^2(t)\rangle$. It obtains contributions from all Fourier modes of Φ:

$$\langle\Phi^2(t)\rangle=\int_{k_i}^{k_t}\frac{k^3}{2\pi^2}|\Phi_k(t)|^2\frac{dk}{k}, \tag{18}$$

where $k_i=a_iH_i$ and $k_t=a(t)H(t)$ are infrared and ultraviolet cutoffs, respectively. The infrared cutoff is given by the length scale above which there are no fluctuations. This is in turn determined by the cosmological model. If our Universe results from a period of inflation, then k_i is given by the comoving wave number corresponding to the Hubble radius at the beginning of inflation. The ultraviolet cutoff is a more tricky issue. We will return to a discussion of this point in [19]. For the moment, let us remark that the phase space of ultraviolet modes increases more slowly in an accelerating Universe than the phase space of infrared modes.

There are two reasonable choices for the ultraviolet cutoff: either at constant physical or at constant comoving wave number k. The first prescription is more physical. In an exponentially expanding background, the resulting phase space of ultraviolet modes is constant, whereas the phase space of infrared modes grows rapidly as the wavelengths of modes are stretched exponentially to become larger than the Hubble radius. For quintessence models with $p > -\rho$, the ultraviolet phase space does grow slowly, but as long as the equation of state of quintessence does not differ too much from $p = -\rho$, the growth of the infrared phase space will exceed the growth of the ultraviolet phase space. The problem with this first way of setting the ultraviolet cutoff is that it requires the continuous creation of modes at the ultraviolet cutoff frequency, and this is hard to reconcile with unitarity. This problem is avoided if the ultraviolet cutoff is set at a constant comoving wave number. In this case, the phase space of ultraviolet modes always decreases in an accelerating background cosmology because the Hubble radius is decreasing in comoving coordinates.

We conclude that, once the ultraviolet cutoff is set at some time (e.g. the present time t_0), the ultraviolet terms will be well controlled at all future times. In the following, in evaluating the strength of the back reaction at time t, we will restrict our attention to the effect of infrared modes, and hence we will only consider the contribution to Eq. (18) of modes with a wave number smaller than the value $k_{H(t)}$ corresponding to the Hubble radius at time t.

III. SPECTRUM OF COSMOLOGICAL FLUCTUATIONS

In this section we will estimate the strength of the back reaction for two classes of quintessence models, assuming that the spectrum of cosmological fluctuations is normalized by the recent observations of cosmic microwave background (CMB) anisotropies, and that the spectrum is either scale invariant or given by a slight tilt. We will find that in the former case back reaction is always small, whereas it can become large and end the period of quintessence domination in the latter case.

To apply the formulas of Sec. II we need to relate the value of Φ_k during the period of quintessence domination, from now on denoted by Φ_{Qk}, to the corresponding value Φ_{mk} at a redshift just before the quintessence begins to dominate, i.e. in the matter-dominated phase. For simplicity,we will take this time to be the present time t_0.

It is convenient to consider separately modes which are outside the Hubble radius today, and modes which are outside the Hubble radius at the time $t > t_0$, but inside at the present time. For the former modes we can use the conservation [22–24] of the Bardeen parameter

$$\zeta \equiv \frac{2}{3}\frac{\Phi + H^{-1}\dot{\Phi}}{1+w} + \Phi, \tag{19}$$

to relate the values of $\Phi_{Qk}(t)$ and $\Phi_{mk}(t_0)$. Here, $w = p/\rho$ is a measure of the equation of state of the background. Thus,

$$\frac{2}{3}\frac{\Phi_{mk} + H_m^{-1}\dot{\Phi}_{mk}}{1+w_m} + \Phi_{mk} = \frac{2}{3}\frac{\Phi_{Qk} + H_Q^{-1}\dot{\Phi}_{Qk}}{1+w_Q} + \Phi_{Qk}, \tag{20}$$

where the subscripts m and Q denote whether the quantity is evaluated during the epoch of matter domination or during quintessence domination.

During the matter dominated era, $w_m = 0$ and $\dot{\Phi}_{mk} = 0$, so the left-hand side of (20) is $5\Phi_{mk}/3$. During the slow-roll period of quintessence domination, we have [from Eqs. (7) and (12)]

$$H_Q^{-1}\dot{\Phi}_k = -A_k^{sup}\frac{m_{pl}^4}{(8\pi)^2}\left(\frac{V'}{V}\right)^2\left[\frac{V''}{V} - \left(\frac{V'}{V}\right)^2\right], \tag{21}$$

where the subscript "*sup*" indicates that the mode is outside the present Hubble radius. In addition,

$$1 + w_Q = \frac{\dot{Q}^2}{V} = \frac{m_{pl}^2}{24\pi}\left(\frac{V'}{V}\right)^2. \tag{22}$$

Combining Eqs. (11), (20), (21), and (22), we have

$$\frac{5}{3}\Phi_{mk} = A_k^{sup}(1 + 5\epsilon - 2\eta), \tag{23}$$

making use of the two slow-roll parameters,

$$\epsilon \equiv \frac{m_{pl}^2}{16\pi}\left(\frac{V'}{V}\right)^2 \ll 1, \tag{24}$$

$$\eta \equiv \frac{m_{pl}^2}{8\pi}\frac{V''}{V} \ll 1. \tag{25}$$

Thus, we have

$$A_k^{sup} \simeq 5\Phi_{mk}/3 \sim \Phi_{mk}. \tag{26}$$

We see that the amplitude of Φ_k during the period of quintessence domination is suppressed by the slow-roll parameter ϵ compared to the corresponding value in the matter-dominated era.

For the modes which are at the present time t_0 inside the Hubble radius, but exit the Hubble radius before time t, we use the fact that while on scales smaller than the Hubble radius the dominant mode of Φ scales as

$$\Phi_{Qk} \propto \dot{Q}_k \propto \frac{V'}{3H_Q} \tag{27}$$

[see Eq. (6.54) of [21]], while on scales larger than the Hubble radius

$$\Phi_{Qk} \propto \left(\frac{V'}{V}\right)^2 \tag{28}$$

[see Eq. (11)]. This allows us to express Φ_{Qk} in terms of the values of the potential and its derivatives at the times t_0, $t_H(k)$ and t, where $t_H(k)$ is the time when the scale k crosses the Hubble radius, and is given by $a(t_H(k))H = k$,

$$\begin{aligned}\Phi_{Qk} &= \frac{1}{6\sqrt{2\pi}m_{pl}}\left(\frac{V^{3/2}}{V'}\right)_{t_H(k)}\frac{V'^2}{V^2}\Phi_{mk}\left(\frac{1}{6\sqrt{2\pi}m_{pl}}\frac{V_0'}{V_0^{1/2}}\right)^{-1}\\ &= \Phi_{mk}\frac{V_0^{1/2}}{V_0'}\left(\frac{V^{3/2}}{V'}\right)_{t_H(k)}\frac{V'^2}{V^2}\\ &\equiv A_k^{sub}\frac{m_{pl}^2}{16\pi}\frac{V'^2}{V^2},\end{aligned} \tag{29}$$

where

$$A_k^{sub} \equiv \frac{16\pi}{m_{pl}^2}\frac{V_0^{1/2}}{V_0'}\left(\frac{V^{3/2}}{V'}\right)_{t_H(k)}\Phi_{mk} \tag{30}$$

is introduced in accord with the notation of Eq. (11). Note that since the quintessence field is a scalar field, we were able to use equations for cosmological perturbations derived for a scalar field-dominated equation of state.

In the following, we will evaluate the two-point function (18) for times $t \gg t_0$, making use of the amplitudes of Φ_{Qk} derived above. We will consider two potentials commonly used for quintessence. First, we will assume that the spectrum of fluctuations at the present time is scale invariant. Later, we will relax this assumption and consider blue spectra.

A. Scale-invariant spectrum

If the present spectrum of fluctuations is scale invariant, then

$$P_m(k) \equiv \frac{k^3}{2\pi^2}|\Phi_{mk}|^2 = C, \tag{31}$$

where the constant C can be fixed by the amplitude of CMB anisotropies on large angular scales measured by the Cosmic Background Explorer (COBE) [25]:

$$\delta_H \simeq 10^{-5} \simeq P_m(k)^{1/2}_{k=aH_m} = C^{1/2}. \tag{32}$$

Substituting Eqs. (11), (26), (30), (31) and (32) into Eq. (18), we finally arrive at

$$\begin{aligned}\langle\Phi_Q^2(t)\rangle &\simeq \int_{k_i}^{k_0} C\left(\frac{m_{pl}^2}{16\pi}\right)^2\frac{V'^4}{V^4}\frac{dk}{k}\\ &\quad + \int_{k_0}^{k_t} C\frac{V_0}{V_0'^2}\left(\frac{V^3}{V'^2}\right)_{t_{k=aH_Q}}\frac{V'^4}{V^4}\frac{dk}{k}\\ &= C\epsilon^2\ln\frac{a_0H_0}{a_iH_i} + C\frac{V_0}{V_0'^2}\frac{V'^4}{V^4}\int_{k_0}^{k_t}\left(\frac{V^3}{V'^2}\right)_{t_H(k)} d\ln k.\end{aligned} \tag{33}$$

Replacing the integral over k by the integral over $a(t_H(k))$ we obtain

$$\langle\Phi_Q^2(t)\rangle \simeq C\epsilon^2\ln\frac{a_0H_0}{a_iH_i} + C\frac{V_0}{V_0'^2}\frac{V'^4}{V^4}\int_{a_0}^{a_t}\left(\frac{V^3}{V'^2}\right)d\ln a, \tag{34}$$

where we have neglected $d\ln H_Q = -\epsilon d\ln a$.

The slow-rolling approximation gives

$$\begin{aligned}a(t) &= a_0\exp\left[-\frac{8\pi}{m_{pl}^2}\int_{Q_0}^{Q}\frac{V}{V'}dQ\right],\\ d\ln a &= -\frac{8\pi}{m_{pl}^2}\frac{V}{V'}dQ.\end{aligned} \tag{35}$$

Using these relations, the expression (34) can be simplified, yielding

$$\langle\Phi_Q^2(t)\rangle \simeq C\epsilon^2\ln\frac{a_0H_0}{a_iH_i} - C\frac{8\pi}{m_{pl}^2}\frac{V_0}{V_0'^2}\frac{V'^4}{V^4}\int_{Q_0}^{Q}\left(\frac{V^4}{V'^3}\right)dQ.$$

The first of the two quintessence models for which we will evaluate the effect of the back reaction is

$$V = BQ^{-\alpha} \tag{36}$$

with B being a constant and $\alpha > 1$. First, note that for this potential, the first term in Eqs. (14) and (15) dominates and determines the equation of state of the effective energy-momentum tensor of the back reaction. In particular, it follows that the equation of state is that of a *negative* cosmological constant, as in the case of the back reaction in inflationary cosmology [17,18]. Thus, we see that the effects of the back reaction oppose the quintessence-driven acceleration. Inserting Eq. (36) into Eq. (33) yields

$$\langle\Phi_Q^2(t)\rangle = C\frac{1}{\pi^2}\frac{m_{pl}^4}{Q^4}\ln\frac{a_0H_0}{a_iH_i} + C\frac{2\pi}{m_{pl}^2}\frac{Q_0^6}{Q^4}\ln\frac{Q}{Q_0}, \quad \text{for} \quad \alpha = 4, \tag{37}$$

$$\langle\Phi_Q^2(t)\rangle = C\frac{\alpha^4}{256\pi^2}\frac{m_{pl}^4}{Q^4}\ln\frac{a_0H_0}{a_iH_i} + C\frac{8\pi}{\alpha(4-\alpha)}\frac{Q_0^2}{m_{pl}^2}\left(\frac{Q_0^\alpha}{Q^\alpha}-\frac{Q_0^4}{Q^4}\right) \quad \text{for} \quad \alpha\neq 4, \tag{38}$$

where $Q_0 = Q(t_0)$. Substituting Eq. (36) into Eq. (17), we have

$$\frac{\rho_{br}+3p_{br}}{\rho_{bg}+3p_{bg}} \simeq \left[2\left(\frac{1}{\alpha}-1\right)+O\left(\left(\frac{m_{pl}}{Q}\right)^2\right)\right]\langle\Phi_Q^2(t)\rangle. \tag{39}$$

In this model, $Q_0 \sim m_{pl}$, and Q increases with time. It can be seen from the combination of Eqs. (37), (38) and (39) that the back reaction is very small now and will be smaller in the future. Notice that the contribution of the first term in Eqs. (37) and (38) is generally small in this case. For example, if we assume the inflation started at the grand unified theory (GUT) scale and the e-folding number is 70, $\ln(a_0H_0/k_i)$ is about $O(10^1)$. When $Q \gg m_{pl}$, the first term can be neglected compared with the second term in Eqs. (37) and (38).

The second model is

$$V = De^{-\lambda Q/m_{pl}} \tag{40}$$

with D and λ constants. In this model, $\epsilon = \lambda^2/16\pi \ll 1$, $\eta = \lambda^2/8\pi \ll 1$. It can easily be verified that also for this potential the first term on the right-hand side of Eqs. (14) and (15) dominates, and once again leads to an equation of state of the effective energy-momentum tensor of the back reaction which acts like a negative cosmological constant and hence opposes the quintessence-driven acceleration. After some straightforward calculation we have

$$\frac{\rho_{br}+3p_{br}}{\rho_{bg}+3p_{bg}} \simeq -C\left(2+\frac{\lambda^2}{4\pi}\right) \times\left[\frac{\lambda^4}{256\pi^2}\ln\frac{a_0H_0}{a_iH_i}+\frac{8\pi}{\lambda^2}\left(1-e^{-\lambda(Q-Q_0)/m_{pl}}\right)\right] \simeq -C\frac{16\pi}{\lambda^2}\left(1-e^{-\lambda(Q-Q_0)/m_{pl}}\right). \tag{41}$$

One can see that for a reasonable value of λ [26], the back reaction in this model is very small too.

B. Blue perturbation spectra

In the above we assumed that the spectrum for the gravitational potential during the matter-dominated era $P_m(k)$ is exactly scale invariant. However, if the primordial spectrum has a slight blue tilt, the conclusion will change dramatically. Thus, we now assume that

$$P_m(k) = C\left(\frac{k}{k_{COBE}}\right)^n, \tag{42}$$

with $0 < n < 0.1$, the upper bound being set by the joint analysis of [27] of the Maxima-1, Boomerang and COBE cosmic microwave anisotropy results. The upper bound corresponds to the one sigma statistical error. Including the estimate of [27] of the systematic errors would increase the bound to $n < 0.27$. In the above, $k_{COBE} \simeq 7.5 a_0 H_0$. The constant C is again given by Eq. (32). Straightforward calculations yield

$$\langle\Phi_Q^2(t)\rangle \simeq \frac{C\epsilon^2}{7.5^n n}\left[1-\left(\frac{a_iH_i}{a_0H_0}\right)^n\right] - \frac{C}{7.5^n}\frac{8\pi}{m_{pl}^2}\frac{V_0^{1-n/2}}{V_0'^2}\frac{V'^4}{V^4} \times\int_{Q_0}^{Q}\frac{V^{4+n/2}}{V'^3}\left(e^{-(8\pi n/m_{pl}^2)\int_{Q_0}^{Q}(V/V')dQ}\right)dQ. \tag{43}$$

As argued in the above, the first term is generally small (notice n is small which also implies that $7.5^n \simeq 1$).

For the first model (36),we have

$$\langle\Phi_Q^2(t)\rangle \simeq C\left(\frac{4\pi(-n)}{\alpha m_{pl}^2}\right)^{-1+[(1+n/2)\alpha/2]}\frac{1}{-n} \times Q_0^{2+(1+n/2)\alpha}e^{-4\pi n Q_0^2/\alpha m_{pl}^2}Q^{-4} \times\Gamma\left(2-\frac{(1+n/2)\alpha}{2}, -\frac{4\pi n}{\alpha m_{pl}^2}Q_0^2, -\frac{4\pi n}{\alpha m_{pl}^2}Q^2\right), \tag{44}$$

where $\Gamma(a,z_0,z_1)$ is the generalized incomplete gamma function:

$$\Gamma(a,z_0,z_1) = \int_{z_0}^{z_1}x^{a-1}e^{-x}dx = \frac{e^{-z_0}z_0^a}{\Gamma(1-a)}\int_0^\infty\frac{e^{-x}x^{-a}}{z_0+x}dx - \frac{e^{-z_1}z_1^a}{\Gamma(1-a)} \times\int_0^\infty\frac{e^{-x}x^{-a}}{z_1+x}dx. \tag{45}$$

From Eqs. (44) and (45) we can see that the back reaction will become important and terminate the acceleration of the Universe driven by the background of quintessence when Q becomes large enough, because we approximately have

$$\langle\Phi_Q^2(t)\rangle \propto \frac{e^{4\pi n Q^2/\alpha m_{pl}^2}}{Q^{2+(1+n/2)\alpha}}. \tag{46}$$

For the second model (40) we have

$$\langle\Phi_Q^2(t)\rangle \simeq C\frac{8\pi}{\lambda m_{pl}}(Q-Q_0), \tag{47}$$

when $1+n[\frac{1}{2}-(1/\eta)]=0$, and

$$\langle \Phi_Q^2(t)\rangle \simeq \frac{C/\eta}{1+n\left(\frac{1}{2}-\frac{1}{\eta}\right)} \times (1-e^{-(\lambda/m_{pl})\{1+n[(1/2)(1/\eta)]\}(Q-Q_0)}). \tag{48}$$

when $1+n[\frac{1}{2}-(1/\eta)]\neq 0$. In the first case, the back reaction will terminate the acceleration of the Universe when $Q \sim 10^{10}\sqrt{\eta/8\pi}m_{pl}$. In the second case, for $n>2\eta/(2-\eta)$ the acceleration will stop when

$$Q\sim \sqrt{\frac{\eta}{8\pi}}\frac{\ln\{10^{10}[n-(1+n/2)\eta]+1\}}{n-(1+n/2)\eta}m_{pl}+Q_0. \tag{49}$$

Recall that the slow-rolling parameter η refers to the dynamics of quintessence whereas the index n refers to the spectrum of fluctuations produced during inflation. Therefore, n and η are independent, and our constraint $n>2\eta/(2-\eta)$ for back reaction to be important does not represent a severe fine tuning.

Note that for $n<2\eta/(2-\eta)$, similar to Eq. (41), the back reaction is generally still small unless the difference between the parameters n and η is fine tuned a lot; for example, if

$$\frac{\eta-10^{-10}}{1-\eta/2}<n<\frac{\eta}{1-\eta/2}, \tag{50}$$

the contribution coming from the back reaction will terminate the accelerating expansion of the Universe in the distant future.

IV. DISCUSSION

We have demonstrated that the gravitational back reaction of cosmological perturbations can lead to a termination of the period of quintessence domination, in the same way it can lead to a termination of a period of inflation [17,18]. In this paper we have considered the effect of infrared modes, i.e. modes whose wavelength at time t (when back reaction is evaluated) is larger than the Hubble radius.

Specifically, we assume a COBE normalized spectrum of cosmological fluctuations at the present time t_0. We consider both a scale-invariant spectrum and spectra with slight blue tilts. We study two classes of quintessence potentials, given by Eqs. (36) and (40). The most important contribution to the back reaction comes from modes with $k_0<k<k_H(t)$, where k_0 is the wave number corresponding to the present Hubble radius, and $k_H(t)$ the wave number corresponding to the Hubble radius at time t. If the spectrum has a blue tilt, it is the upper limit which dominates. We find that for spectra with a sufficiently large blue tilt, back reaction will terminate the period of quintessence domination. The effective energy-momentum tensor which describes the back reaction has the form of a negative cosmological constant and hence opposes the quintessence-driven acceleration.

Since the dominant contribution in our analysis comes from the largest values of k we consider, it is to be expected that the contribution to the back reaction from ultraviolet modes (modes whose wavelength at time t is smaller than the Hubble radius) may dominate over the effects calculated here. We will return to this issue in [19]. The point of our analysis was to show that back reaction cannot be neglected when considering the future of a Universe dominated at the present time by quintessence.

The wavelength of modes which dominate the contribution to our effect was smaller than the Hubble radius at t_0. Hence, these modes are subject to the usual nonlinear growth of cosmological fluctuations in the matter-dominated phase. This is another effect which has not been taken into account here (but will be in [19]). However, it is reasonable to expect that nonlinear effects will enhance the amplitude of the fluctuations and hence increase the back-reaction effects.

Obviously, if the spectrum of fluctuations is so close to scale invariance so that the back reaction only becomes important in the very distant future, then the modes which will contribute in this case had a microphysical wavelength at the present time, and we are faced with the true ultraviolet problem of field theory.

It has recently been claimed [8,9] that the presence of future horizons in quintessence models poses a problem in trying to reconcile string theory and quintessence. In models where our back-reaction effect becomes important, this problem may well disappear (without the need to have to introduce extra physics such as additional scalar fields).

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